

# OPEN-CONSTRUCTIBLE FUNCTIONS

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**ABSTRACT.** We prove that if a continuous function  $f : X \rightarrow f(X)$  takes open sets into elements of the Boolean algebra generated by open and closed subsets in  $f(X)$ , then there exist  $X_n \subset X$ ,  $(n \in \omega)$  such that  $f$  is open on every  $X_n$  and  $f(X_n)$  cover  $f(X)$ .

## 1. Introduction

All spaces are supposed to be separable and metrizable.

Recall that *constructible* sets are finite unions of locally closed sets (a set is *locally closed* if it is the intersection of an open set and a closed set). In fact, the constructible sets are precisely Boolean algebras generated by open sets and closed sets (see, for example, [1], [6]).

In algebraic geometry, a constructible set is any zero set of a system of polynomial equations and inequations.

A function  $f$  is said to be *open-constructible* if  $f$  takes open sets into constructible sets.

In particular,  $f$  is *open* if  $f$  takes open sets into open sets.

The following Theorem 1 gives an extension of some recent results of the author (see, for example, [4]) that provides a generalization of the classical theorems on the preservation of Borel classes to the setting of open-constructible functions [5].

**Theorem 1.** *For every continuous, open-constructible function  $f : X \rightarrow f(X)$  of a separable metric space  $X$  there exist constructible subsets  $X_n \subset X$  such that every restriction  $f|_{X_n}$  is open and  $f(X) = \bigcup_{n \in \omega} f(X_n)$ .*

The proof is based on the concept of a homogeneously non-open function defined on a subspace of the Cantor set  $\mathbf{C}$ .

We will consider for every  $n = 1, 2, \dots$  a cover  $\gamma_n$  of the Cantor set  $\mathbf{C}$  consisting of pairwise disjoint clopen sets  $V_{i_n}, i_n = 1, \dots, 2^n$ , such that  $\text{diam} V_{i_n} = \frac{1}{3^n}$  and each element of  $\gamma_{n+1}$  lies in some element of  $\gamma_n$ .

## 2. Decomposing functions into open and homogeneously non-open

In this section, we assume that  $X \subset \mathbf{C}$ .

Let  $\mathcal{P}$  be a class of functions  $f : X \rightarrow f(X)$ . We say that  $f$  is *homogeneously*  $\mathcal{P}$  if, for every clopen subset  $U \subset X$ , the restriction  $f|_U$  belongs to  $\mathcal{P}$ .

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For example, if  $\mathcal{P}$  is the class of open (resp., closed or continuous) functions, then every  $f \in \mathcal{P}$  is homogeneously open (resp., homogeneously closed or homogeneously continuous).

We say that  $f : X \rightarrow f(X)$  is *homogeneously non- $\mathcal{P}$*  if there exists a nonempty clopen set  $U \subset X$  such that the restriction  $f|U$  does not belong to  $\mathcal{P}$ .

In particular, a function  $f$  is called *homogeneously non-open* if there is a nonempty clopen set  $U \subset X$  such that  $f|U$  is not open at a point  $x \in U$ .

**Proposition 2.1.** *Let  $f \in \mathcal{P}$  and for every open  $O \subset X$  the restrictions  $f|f^{-1}f(O)$  and  $f|(X \setminus f^{-1}f(O))$  belong to  $\mathcal{P}$ . Then there exist the pairwise disjoint sets  $X_n \subset X, (n \in \omega)$  such that*

- every restriction  $f|X_n$  belongs to  $\mathcal{P}$ ;
- the restriction  $f|Z$ , where  $Z = X \setminus f^{-1}f(\bigcup_n X_n)$ , is homogeneously non- $\mathcal{P}$ ;
- every  $X_n$  can be written in the form  $V \cap f^{-1}f(U)$ , where  $V$  and  $U$  are open subsets in  $X$ .

*Proof.* If there exists a nonempty clopen set  $V_{i_{n(1)}} \in \gamma_{n(1)}$  for which  $f|V_{i_{n(1)}} \in \mathcal{P}$ , then consider the set

$$X_1 = X \setminus f^{-1}f(V_{i_{n(1)}})$$

and

$$f_1 = f|X_1.$$

Now, we apply the construction that we have applied to the set  $X$  inductively. We thus obtain

$$X_{k+1} = X_k \setminus f_k^{-1}f_k(V_{i_{n(k)}} \cap X_k)$$

with

$$f_{k+1} = f|X_{k+1}.$$

For limit ordinal  $\alpha$ , define

$$X_\alpha = X \setminus \bigcup_{\beta < \alpha} X_\beta$$

with

$$f_\alpha = f|X_\alpha.$$

If the process continues, then we get a chain of spaces

$$X \supset X_1 \supset \dots \supset X_\gamma \supset \dots$$

As is known, the sequence of open sets  $U_\gamma = \bigcup_{\beta < \gamma} V_{i_{n(\beta)}}$  in a separable metric space stabilizes at some  $\gamma_0 < \omega_1$ .

$$Z = X_{\gamma_0} = X_{\gamma_0+1} = \dots$$

Writing  $X_n$  instead  $X_\gamma$ , we obtain

$$Z = X \setminus f^{-1}f\left(\bigcup_n X_n\right) = X \setminus f^{-1}f(U_{\gamma_0}).$$

Obviously,  $g = f|Z$  is homogeneously non- $\mathcal{P}$ . □

It is easy to see that the classes  $\mathcal{P}$  of open (closed, continuous) functions satisfy the hypotheses of Proposition 2.1, and we have:

**Corollary 1.** *For each function  $f : X \rightarrow f(X)$ , with  $X \subset \mathcal{C}$ , there exist the pairwise disjoint sets  $X_n \subset X, (n \in \omega)$  such that*

- *every restriction  $f|X_n$  is open;*
- *the restriction  $f|Z$ , where  $Z = X \setminus f^{-1}f(\bigcup_{n \in \omega} X_n)$ , is homogeneously non-open;*
- *if  $f$  is continuous and open-constructible, then  $X_n$  are constructible subsets of  $X$ .*

□

Our goal is to prove that  $Z = \emptyset$  for every open-constructible function  $f$ .

The proof consist in the construction of compact sets  $S_{2m}(y) \subset g(Z)$  and open sets  $\mathfrak{T}_m$  that satisfy the hypotheses of the following lemma.

### 3. Main Lemma

Consider the space  $Z$  from Corollary 1.

**Lemma 1.** *For every natural number  $m = 1, 2, \dots$ , there exists an open subset  $\mathfrak{T}_m$  in  $X$  and a compact set  $S_{2m}(y) \subset g(Z)$  such that  $S_{2m}(y) \setminus g(\mathfrak{T}_m \cap Z)$  is not a union of  $m$  subsets that are locally closed in  $g(Z)$ .*

Proof. **Step 1.** Denote by

$$S_1(y) = \{y\} \cup \{y_{k_1}\}$$

a sequence  $y_{k_1} \rightarrow y$  ( $k_1 \in \omega$ ) with limit point  $y$ .

It is easy to prove that the function  $g : Z \rightarrow g(Z)$  is open if and only if

$$g^{-1}(y_i) \cap O(z) \neq \emptyset$$

for every  $S_1(y) \subset g(Z)$ , every point  $z \in g^{-1}(y)$  and every open neighborhood  $O(z)$  of  $z$ .

Let us consider an arbitrary point  $x \in Z$  and all sets  $V_{i_n} \in \gamma_n, (n \in \omega)$  containing  $x$ . Since  $g$  is not open on every set  $V_{i_n} \cap Z$ , there exists a sequence (without repetitions)  $x_{k_1} \rightarrow x$  such that  $g$  is not open at  $x_{k_1}$ ; i.e., for every  $k_1 \in \omega$  there exist a sequence  $y_{k_1 k_2} \rightarrow y_{k_1} = g(x_{k_1}) \rightarrow y = g(x)$  and a set  $V_{i_n}(x_{k_1})$  such that

$$(1) \quad V_{i_n}(x_{k_1}) \cap g^{-1}(y_{k_1 k_2}) = \emptyset$$

for every  $k_2 \in \omega$ .

Take the pairwise disjoint open in  $g(Z)$  neighborhoods  $O(y_{k_1}) \not\ni y$  of points  $y_{k_1}$ .

Since  $f$  is continuous, we can assume that

$$(2) \quad V_{i_n}(x_{k_1}) \cap Z \subset g^{-1}(O(y_{k_1})).$$

Denote

$$\mathfrak{T}_1 = \bigcup_{k_1 \in \omega} V_{i_n}(x_{k_1})$$

and

$$S_2(y) = \{y\} \cup \{y_{k_1}\} \cup \{y_{k_1 k_2}\}.$$

Then  $\mathfrak{T}_1$  is open in  $X$ , and, by conditions (1) and (2),

$$S_2(y) \setminus g(\mathfrak{T}_1 \cap Z) = \{y\} \cup \{y_{k_1 k_2}\}.$$

To prove that  $\{y\} \cup \{y_{k_1 k_2}\}$  is not a union of an open and a closed set in  $S_2(y)$ , suppose the opposite:

$$(3) \quad \{y\} \cup \{y_{k_1 k_2}\} = O_1 \cap F_1,$$

where  $O_1$  is an open and  $F_1$  is a closed subset of  $S_2(y)$ .

Since  $y \in O_1$ , there is a number  $N_1$  such that  $y_{k_1} \in O_1$  if  $k_1 > N_1$ .

Since  $y \in F_1$ , there is a number  $N_2$  such that  $y_{k_1 k_2} \in F_1$  for  $k_1 \geq N_2$  only for finitely many  $k_2$ ; hence, there exists a  $y_{k_1 k_2} \notin O_1 \cap F_1$  that contradicts (3). □

Suppose that, at the step  $m$ , we have already constructed

- the countable compact set

$$S_{2m}(y) = \{y\} \cup \{y_{k_1}\} \cup \{y_{k_1 k_2}\} \cup \cdots \cup \{y_{k_1 \dots k_{2m}}\}$$

with

$$y_{k_1 \dots k_{2m}} \rightarrow y_{k_1 \dots k_{2m-1}} \rightarrow \cdots y_{k_1 k_2} \rightarrow y_{k_1} \rightarrow y = g(x);$$

- the pairwise disjoint neighborhoods  $O(y_{k_1 \dots k_{2m-1}}) \not\supset y_{k_1 \dots k_{2m-2}}$  of points

$$y_{k_1 \dots k_{2m-1}} = g(x_{k_1 \dots k_{2m-1}});$$

- the clopen sets  $V_{i_n}(x_{k_1 \dots k_{2m-1}})$  with properties:

$$V_{i_n}(x_{k_1 \dots k_{2m-1}}) \cap Z \subset g^{-1}(O(y_{k_1 \dots k_{2m-1}}))$$

and

$$(4) \quad V_{i_n}(x_{k_1 \dots k_{2m-1}}) \cap g^{-1}(y_{k_1 \dots k_{2m}}) = \emptyset;$$

- the set

$$\mathfrak{T}_m = \bigcup_{k_{2m-1} \in \omega} V_{i_n}(x_{k_1 \dots k_{2m-1}}) \cup \mathfrak{T}_{m-1}$$

that is open in  $X$  and such that

$$S_{2m}(y) \setminus g(\mathfrak{T}_m) = \{y\} \cup \{y_{k_1 k_2}\} \cup \cdots \cup \{y_{k_1 \dots k_{2m}}\}$$

is not a union of  $m$  locally closed sets.

**Step  $m+1$ .** By (4), every point  $y_{k_1 \dots k_{2m}} \in S_{2m}(y)$  is the image of a point

$$x_{k_1 \dots k_{2m}} \in g^{-1}(y_{k_1 \dots k_{2m}}) \setminus V_{i_n}(x_{k_1 \dots k_{2m-1}}),$$

and, since the set  $V_{i_n}(x_{k_1 \dots k_{2m-1}})$  is clopen<sup>1</sup>, there is a clopen set  $V_{i_n}(x_{k_1 \dots k_{2m}})$  such that

$$V_{i_n}(x_{k_1 \dots k_{2m}}) \cap V_{i_n}(x_{k_1 \dots k_{2m-1}}) = \emptyset.$$

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<sup>1</sup>In this place the assumption  $\dim X = 0$  is essential: the requirement " $V_{i_n}(x_{k_1 \dots k_{2m-1}})$  is clopen" cannot be replaced by the weaker requirement " $V_{i_n}(x_{k_1 \dots k_{2m-1}})$  is open".

It is easily seen that we can find pairwise disjoint open sets  $O(y_{k_1 \dots k_{2m}}) \subset O(y_{k_1 \dots k_{2m-1}})$  and assume that

$$V_{i_n}(x_{k_1 \dots k_{2m}}) \subset g^{-1}(O(y_{k_1 \dots k_{2m}})).$$

It is clear that the family of all clopen sets  $V_{i_n}(x_{k_1 \dots k_{2m}})$  chosen at an arbitrary step is discrete.

Since  $g$  is homogeneously non-open we can find (see Step 1) for every point  $x_{k_1 \dots k_{2m}}$  a sequence of points

$$x_{k_1 \dots k_{2m} k_{2m+1}} \rightarrow x_{k_1 \dots k_{2m}}$$

and a sequence

$$y_{k_1 \dots k_{2m} k_{2m+2}} \rightarrow y_{k_1 \dots k_{2m+1}} = g(x_{k_1 \dots k_{2m+1}})$$

such that the following conditions holds for every open neighborhood  $O(x_{k_1 \dots k_{2m+1}})$  of  $x_{k_1 \dots k_{2m+1}}$ :

$$O(x_{k_1 \dots k_{2m+1}}) \cap g^{-1}(y_{k_1 \dots k_{2m+2}}) = \emptyset.$$

Define

$$\mathfrak{T}_{m+1} = \mathfrak{T}_m \cup \bigcup_{k_{2m+1} \in \omega} V_{i_n}(x_{k_1 k_2 \dots k_{2m+1}})$$

and

$$S_{2(m+1)}(y) = \{y\} \cup \{y_{k_1}\} \cup \{y_{k_1 k_2}\} \cup \dots \cup \{y_{k_1 \dots k_{2(m+1)}}\}.$$

Finally,

$$S_{2(m+1)}(y) \setminus g(\mathfrak{T}_{m+1}) = \{y\} \cup \{y_{k_1 k_2}\} \cup \dots \cup \{y_{k_1 \dots k_{2(m+1)}}\}$$

is not a union of  $m+1$  locally closed sets in  $S_{2(m+1)}(y)$ .

Indeed, suppose the opposite and take some set  $O_{m_0} \cap F_{m_0}$  that contains the point  $y$ .

If  $O_{m_0} \cap F_{m_0}$  contains infinitely many points  $y_{k_1 k_2} \in S_{2(m+1)}(y)$  for infinitely many subscripts  $k_1$ , then, obviously,  $F_{m_0}$  contains infinitely many points  $y_{k_1}$ , and there exists a  $y_{k_1} \in O_{m_0} \cap F_{m_0}$ , which is impossible. This implies that, for some number  $N_1$ , the set

$$\{y_{N_1 k_2}\} \cup \dots \cup \{y_{N_1 \dots k_{2m+1}}\}$$

is homeomorphic to  $S_{2m}(y)$  and simultaneously is a union of  $m$  locally closed sets, which contradicts the induction hypothesis. This proves Lemma 1.  $\square$

#### 4. Proof of Theorem 1

It is clear that, for  $X \subset \mathbf{C}$ , the conclusion of Theorem 1 follows from Corollary 1 if  $Z = \emptyset$ . Therefore, we only need to prove the following lemma.

**Lemma 2.** *If the function  $f$  in Corollary 1 is open-constructible, then  $Z = \emptyset$ .*

Proof. Let  $y_{k_1} \rightarrow y$  be a sequence with pairwise disjoint neighborhoods  $O(y_{k_1}) \not\supset y$  and  $V_{i_n}(x_{k_1}) \cap Z \subset g^{-1}(O(y_{k_1}))$  (see Step 1).

We can take according to Lemma 1 the open in  $X$  sets  $\mathfrak{T}_{k_1} \subset V_{i_n}(x_{k_1})$  for  $k_1 = 1, 2, \dots$  and compact  $S_{2k_1}(y_{k_1}) \subset O(y_{k_1})$  such that  $S_{2k_1}(y_{k_1}) \setminus g(\mathfrak{T}_{k_1})$  is not a union of  $k_1$  locally closed in  $g(Z)$  subsets.

The union  $\mathfrak{T}_\infty$  of all sets  $\mathfrak{T}_{k_1}$  is open in  $Z$ , and, hence, there exists an open subset  $V$  in  $X$  for which  $V \cap Z = \mathfrak{T}_\infty$ .

By the hypothesis of Theorem 1, the set  $g(V)$  is constructible in  $f(X) \supset g(Z)$ .

It follows from the definition of  $Z$  that  $f^{-1}(g(Z)) = Z$ ; hence,  $g(\mathfrak{T}_\infty) = f(V \cap Z) = f(V) \cap g(Z)$  is a constructible subset of  $g(Z)$ .

This implies that, for some number  $N_0$ , the set  $g(Z) \setminus g(\mathfrak{T}_\infty)$  is a union of  $N_0$  subsets that are locally closed in  $g(Z)$ .

Since, by construction, the compact sets  $S_{2k_1}(y_{k_1}) \subset g(Z)$  are pairwise disjoint for  $k_1 \in \omega$ , this contradicts the induction hypothesis:  $S_{2N_0}(y_{N_0}) \setminus g(\mathfrak{T}_{N_0})$  is not a union of  $N_0$  sets that are locally closed in  $S_{2N_0}(y_{N_0})$ .

Finally, this yields  $Z = \emptyset$ , and Theorem 1 is proved by Corollary 1 for  $X \subset \mathbf{C}$ .

Using standard arguments we can assume that  $X$  is only separable metrizable.

Indeed, it is well known that the separable metric space  $X$  is homeomorphic to a subspace of a Polish space  $\mathbf{K}$  and  $\mathbf{K}$  is the image of the space of irrational numbers  $\mathbf{P} \subset \mathbf{C}$  under a continuous, open function  $h$ .

Thus,  $X$  is the open image of  $h^{-1}(X) \subset \mathbf{P}$  under a continuous and open function  $h$ . Denote by  $F$  the composition  $f \circ h$ . It is easy to see that the restriction

$$F|_{h^{-1}(X)} \rightarrow f(X)$$

is continuous and open-constructible.

By Proposition 2.1 there exists  $T_n = V \cap F^{-1}F(U)$ , where  $V$  and  $U$  are open subsets in  $h^{-1}(X)$  such that every restriction  $F|_{T_n}$  is open.

Denote  $X_n = h(T_n)$ . It is easily seen that  $X_n = f^{-1}F(T_n)$  is constructible in  $X$ , every restriction  $f|_{X_n}$  is open and  $f(X_n)$  cover  $f(X)$ . This finishes the proof of Theorem 1. □

Analogously to definition of resolvable map [2], we say that  $f$  is constructible if it is either open-constructible or closed-constructible.

Then, according to Theorem 1 and [3], every continuous, constructible function  $f : X \rightarrow f(X)$  is countable open or countable closed, i.e., there exist  $X_n \subset X$  such that every restriction  $f|_{X_n}$  is open or closed and  $f(X_n)$  cover  $f(X)$ .

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